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Upscaling transmissivity under radially convergent flow

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Abstract When pumping from a single well in a real (heterogeneous) aquifer, the steady state drawdown at the well is proportional to the pumping rate through a value called the equivalent transmissivity, T_{eq}, which, in short, is the value that best suits Thiem's formula. Equivalent transmissivity is not a local value, but some representative value of a certain area surrounding the well. By means of numerical simulations we study the validity of a recent analytical formula (Sánchez-Vila *et al.*, 1999) that provides a value for T_{eq} which is given as a weighted average of the fluctuations in log T throughout the domain. The formula is found to work acceptably well for mildly heterogeneous T fields.

INTRODUCTION

Most field methods used to estimate transmissivity values rely on the analysis of drawdown under radially convergent flow conditions. In heterogeneous media, flow towards a pumping well is still convergent, but not uniformly radial. If we consider a single well pumping in a real (heterogeneous) aquifer and under steady state flow conditions, the drawdown at the well, s, is directly related to the pumping rate, Q, through a value called the equivalent or upscaled transmissivity, T_{eq}. This value would be the transmissivity assigned to the well location in a simple interpretation of a steady state pumping test (using Thiem's formula).

Equivalent transmissivity is not a local value, but some representative value of a certain area surrounding the well. Recently, Sánchez-Vila *et al.* (1999) provided an analytical formula for T_{eq}, based upon an extension of Thiem's equation. The resulting upscaled value is given as a weighted average of the fluctuations in log T throughout the heterogeneous domain. In this paper we test this analytical formula by means of numerical simulations for a number of heterogeneous transmissivity fields.

THE ANALYTICAL UPSCALING FORMULA

A rigorous definition (although not necessarily the only possible one) of block or equivalent transmissivity, T_{eq}, in a two-dimensional annular domain is the value that suits the following relationship:

$$Q = 2\pi A T_{eq} \tag{1}$$

Q being the pumping rate and $A = (h_e - h_w)/\ln(r_e/r_w)$; h_w and h_e are the measured heads at the inner (r_w) and outer (r_e) radii, respectively. In short, T_{eq} is the value that best suits Thiem's formula.

In general it is not possible to derive an exact formula for T_{eq} in a heterogeneous domain. Trying to overcome this problem, Sánchez-Vila *et al.* (1999) presented an analytical expression for T_{eq} given as a weighted average of the fluctuations in log T throughout the domain. The solution for T_{eq} is based on a series expansion of Q:

$$Q = \sum_{i=0}^{\infty} Q_i\,; \qquad Q_i = \int_0^{2\pi} T_w \left[\frac{\partial}{\partial r} h^{(i)}(r,\theta) \right]_{r=1} \mathrm{d}\theta \tag{2}$$

where T_w is the transmissivity at the well (computed as the exponential of the spatial average log transmissivity at r_w), and each $h^{(i)}$ comes from a series expansion of h. The final solution is:

$$T_{eq} = \frac{1}{2\pi A}\left(Q_0 + Q_1 + Q_{2a} + Q_{2b} + Q_{2c}\right) \tag{3}$$

$$Q_0 = 2\pi A T_w\,; \quad Q_1 = \frac{Q_0}{W}\int_V \frac{Y'(\rho,\phi)}{\rho^2}\mathrm{d}V \;; \quad Q_{2a} = -\frac{Q_0}{2W}\int_V \frac{Y'^2(\rho,\phi)}{\rho^2}\mathrm{d}V$$

$$Q_{2b} = \frac{Q_0}{W^2}\left(\int_V \frac{Y'(\rho,\phi)}{\rho^2}\mathrm{d}V\right)^2;$$

$$Q_{2c} = \frac{Q_0}{W}\sum_{n=1}^{\infty}\int_V\int_V \frac{Y'(\rho,\phi)Y'(\rho^*,\phi^*)}{\rho^2\rho^{*2}} H_n(\rho,\phi,\rho^*,\phi^*)\mathrm{d}V\mathrm{d}V^*$$

with:

$$H_n(\rho,\phi,\rho^*,\phi^*) = \begin{cases} f_n(\rho)g_n(\rho^*)\Lambda_n(\phi,\phi^*) & \text{with } \rho^* > \rho \\ f_n(\rho^*)g_n(\rho)\Lambda_n(\phi,\phi^*) & \text{with } \rho^* \le \rho \end{cases}$$

$$f_n(\rho) = \frac{\rho^n + \rho^{-n}}{R^n - R^{-n}}\,; \quad g_n(\rho) = (R/\rho)^n + (R/\rho)^{-n}\,; \quad \Lambda_n(\phi,\phi^*) = \frac{n}{2\pi}\cos n(\phi - \phi^*)$$

Here, $Y = \log T$ are the point log transmissivity values, $Y'(r,\theta) = \ln\left[T(r,\theta)/T_w\right]$, V represents the annular domain, $\mathrm{d}V = \rho\,\mathrm{d}\rho\,\mathrm{d}\phi$, ρ is the radial distance to the centre of the well, W is a geometric factor ($W = 2\pi \ln R$), and $R = r_e/r_w$. The approach assumes implicitly that each of the Q_i terms is small compared to the preceding ones, so that after a few terms, we are capable of capturing most of the features of Q, and we can drop the higher order terms. A sufficient condition is that heterogeneity is not very strong (i.e. the variance of Y, $\sigma_Y^2 << 1$).

METHODOLOGY FOR THE VERIFICATION OF THE FORMULA

The methodology is as follows: (a) a domain which consists in an annulus with an inner radius equal to the well radius and an outer radius is selected; (b) a single

realization of a heterogeneous Y field is generated in this particular domain; (c) the steady state flow equation is solved under constant head boundary conditions applied to the inner and outer radii by using the finite element code CFLOW (Axness *et al.*, 1998); (d) as an outcome of CFLOW, the value of the discharge at the well, Q, is computed, so that T_{eq} can also be calculated by direct application of Thiem's formula; and (e) the numerically obtained value T_{eq} is compared to the analytically derived value obtained from the upscaling formula, equation (3).

This methodology is applied to a number of fields with predefined heterogeneous structure. It is important to note that we use only one realization per heterogeneous structure, so we do not cast the problem in a geostatistical framework (i.e. we do not take expected values).

To simulate flow in a heterogeneous medium with an annular domain we use the code CFLOW. This is a finite element program especially designed to compute accurately the flow to the well in this particular geometry. The domain is discretized into circular sectors; then a simple change of the Cartesian coordinates (x,y) into $(\log r,\theta)$ changes the shape of the elements into rectangles. Axness *et al.* (1998) show by means of numerical examples that this provides more accurate flow rate values than traditional finite elements working either with Cartesian or polar coordinates.

Now, to obtain T_{eq} by means of equation (3) we need to solve numerically the corresponding integrals appearing in the Q_i terms. For doing so we take advantage of the domain V being partitioned into sectors. The partition $(\Omega_1,\Omega_2,\Omega_3...,\Omega_{NS})$, NS being the number of elements, is such that:

$$(r,\theta)\in\Omega_i \Leftrightarrow \begin{cases} r_1^{(i)} \le r < r_2^{(i)} \\ \theta_1^{(i)} \le \theta < \theta_2^{(i)} \end{cases}$$

and in each sector the log-transmissivity is considered constant:

$$Y'(r,\theta)=Y'^{(i)} \qquad \text{if } (r,\theta)\in\Omega_i$$

Given $Y'(r,\theta)$ in such a way, we can compute the integrals appearing in equation (3) as a sum of a finite number of terms. As an example, we present the computation of Q_1:

$$Q_1=\frac{Q_0}{W}\int_V \frac{Y'(\rho,\phi)}{\rho^2}\mathrm{d}V=\frac{Q_0}{W}\sum_{i=1}^{NS} Y'^{(i)}\left(\theta_2^{(i)}-\theta_1^{(i)}\right)\ln\left(\frac{r_2^{(i)}}{r_1^{(i)}}\right)$$

The remaining terms can be computed in an analogous form. This methodology has been implemented in a computer code, INTEQ (Bayer, 1999).

NUMERICAL SIMULATIONS

We present a total of five different heterogeneous Y fields in which we have applied the methodology previously explained. Here we present a summary of the results, which include for each field a brief description of the generated field and the transmissivity values computed numerically (from CFLOW) and analytically (using equation (3)). In each field we present the curves for T_{eq} as a function of the contrast in the transmissivity values.

Field #1: Radial T field [$T(r)$]

In this field we consider a case where the T value is constant from the well up to a certain radius and constant again (but with a different value) from there to the external radius. This allows checking with an analytical result (T_{eq} should be the harmonic mean of the T values weighted by the square of the distance to the well). In this particular field we consider that the external radius is 50 times the internal one, while the radius in which T changes is 30 r_w.

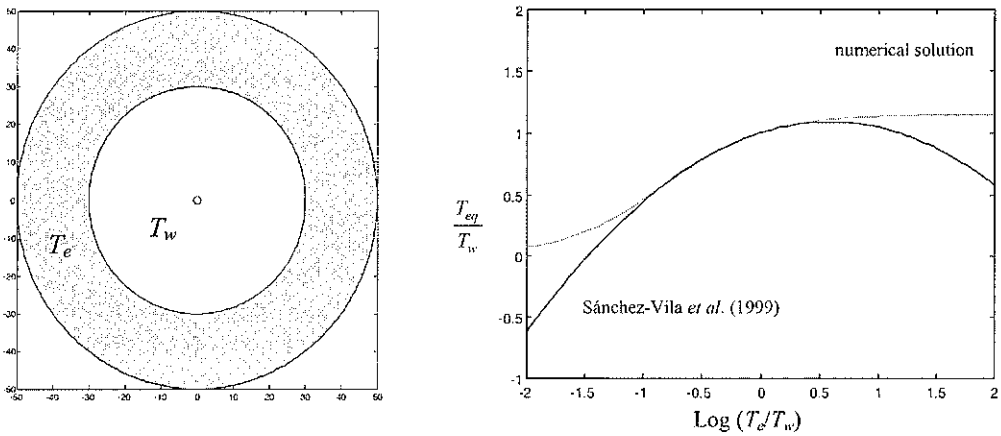

Fig. 1 Left: Field #1 (T radial). Right: Analytical (equation 3) and numerical (CFLOW) values of T_{eq} as a function of the ratio between the two transmissivity values.

Note that the analytical solution is in very good agreement with the real one, up to one order of magnitude contrast in transmissivity. From then on, both solutions diverge, and the analytical formula, being a polynomial approximation, can lead to negative unphysical values.

Field #2: Azimuthal T field [$T(\theta)$]

This particular field was included in Sánchez-Vila *et al.* (1999) as a particular case where the upscaling formula worked particularly well. In this field Y' is given by:

$$Y'(r,\theta) = \begin{cases} 0 & \text{if} \quad 1 < r/r_w < 1.5 \\ \beta\cos(3\theta) & \text{if} \quad 1.5 < r/r_w < 50 \end{cases}$$

For this particular T field (see Fig. 2) the integrals appearing in equation (3) can be calculated analytically in terms of β, so that:

$$\frac{Q_1}{Q_0} = 0; \quad \frac{Q_{2b}}{Q_0} = 0; \quad \frac{Q_{2a}}{Q_0} = -0.224\beta^2; \quad \frac{Q_{2c}}{Q_0} = 0.429\beta^2$$

Note that the β value is a measure of the variance of the log T field, as $\sigma_Y^2 = \beta^2/2$. It is clear from Fig. 2 that the upscaling formula (equation 3) matches the numerical simulations up to $\beta = 4$.

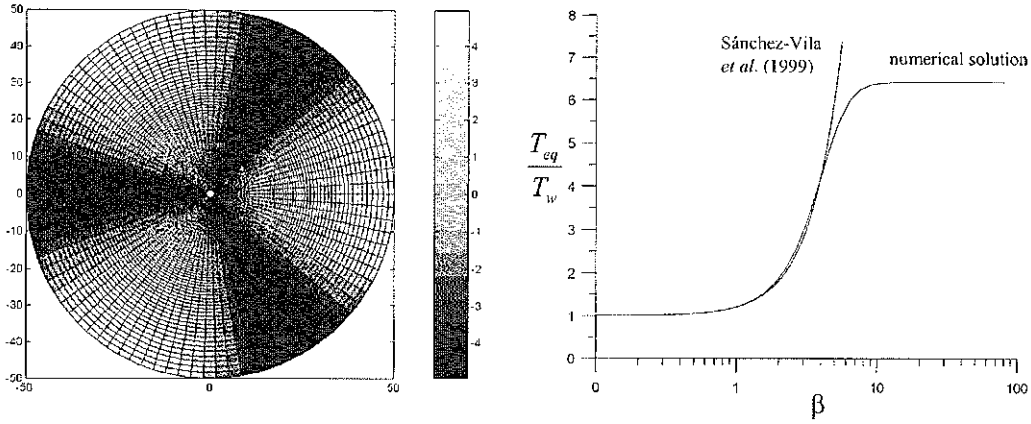

Fig. 2 Left: Field #2 for a value of β = 5. Right: Analytical and numerical T_{eq} values as a function of β.

Field #3: Random *T* field (uncorrelated)

In this case we have generated a large number of values drawn randomly from a normal distribution $N(0,1)$. We have then assigned these values to the different elements in which we have discretized our domain, which is 60 partitions in the radial direction (so that the distances are constant in log r space) and 60 in the azimuthal one (with constant θ). Figure 3 shows the resulting field. This is a degenerated case from the correlated ones, using an integral scale very close to zero.

In any case, it seems that up to $\sigma_Y = 1$ (the standard deviation of log T), the numerically computed T_{eq} value is very close to the exponential of the average log T value at the well, with deviations of less than 4%. This behaviour is well reproduced by the analytical upscaled value, with deviations around the numerically computed value of again 4%.

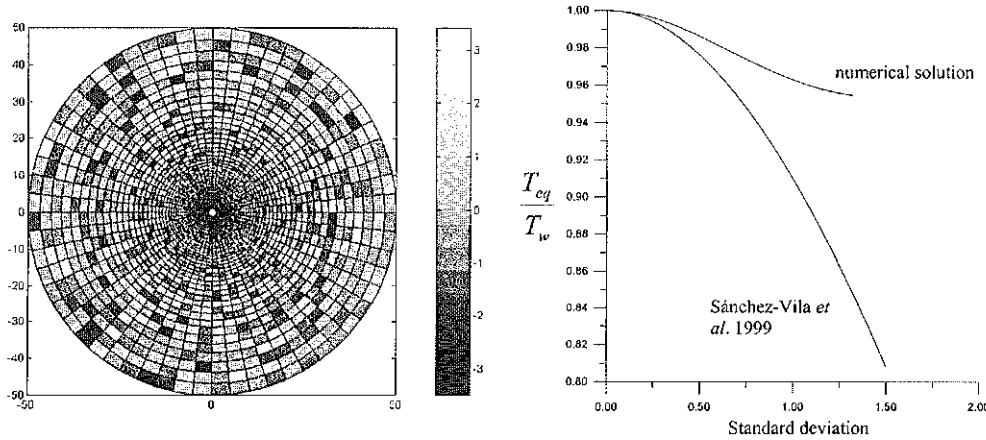

Fig. 3 Left: Field #3 (random). Right: Analytical and numerical T_{eq} values as a function of σ_Y.

An important thing to note here and in the remaining two fields is that in the term Q_{2c} an infinite sum of a double integral appears. The methodology already explained substitutes the integrals by a finite sum, but the infinite sum must still be computed. While this sum has been proven to converge under very general cases (Sánchez-Vila *et al.*, 1999), in this particular field convergence was quite slow and the infinite sum could only be truncated after 400 terms were computed. Figure 4 shows the convergence of the series.

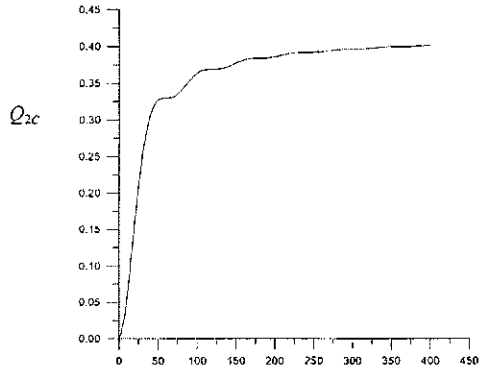

Fig. 4 Convergence of the series in the Q_{2c} term.

Field #4: MultiGaussian *T* field ($I = 5$)

The field was generated using the Turning Bands Method, with an integral scale equal to 5 times the radius of the well (while $r_e = 50r_w$). The resulting field can be seen in Fig. 5.

Again the match is acceptable up to $\sigma_Y = 1$, with deviations of less than 5%.

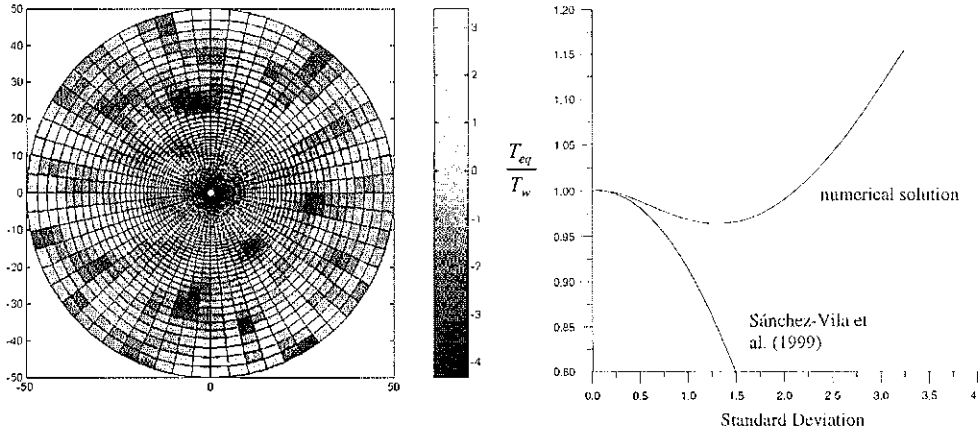

Fig. 5 Left: Field #4. Right: Analytical and numerical T_{eq} values as a function of σ_Y.

Field # 5: MultiGaussian *T* field ($I = 15$)

Now we increase the integral scale up to 15, a very large value compared to the size of the domain (Fig. 6). In this case the match between the analytical and the numerical solutions is acceptable up to standard deviations close to 4.

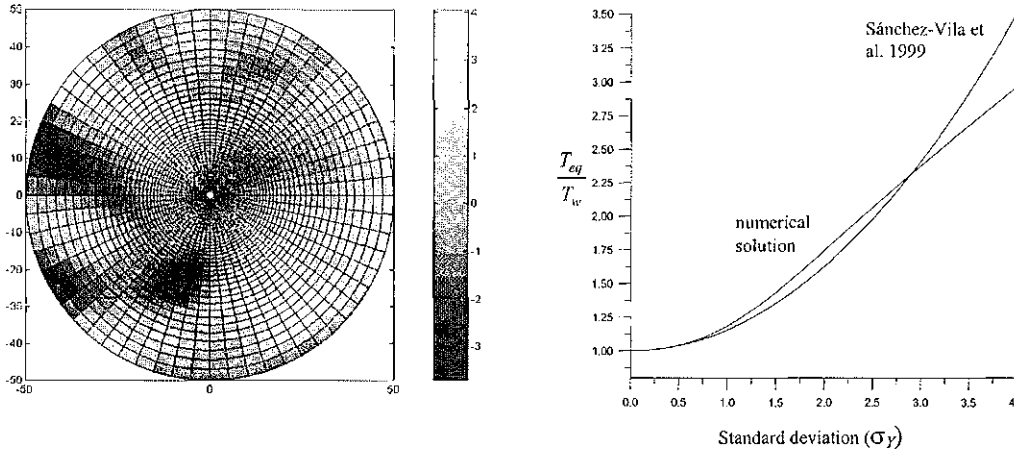

Fig. 6 Left: Field #5. Right: Analytical and numerical T_{eq} values as a function of σ_Y.

CONCLUSIONS

As is well known, transmissivity derived from the interpretation of a pumping test is not only dependent upon local conditions in the area of the well, but it represents an average value over a representative volume of aquifer. In this paper we test an analytical expression for the weighting function involved in the averaging process, which is valid independently of the univariate or multivariate distributions of the *T* field.

We have presented five different transmissivity fields and tested the analytical upscaling formula. In all the cases the performance can be considered acceptable for mildly heterogeneous media. The main problem associated with the solution is the need for numerically evaluating an infinite sum. We have seen at least one case where the converg-ence of the series is very slow, leading to quite a large CPU time demanding method.

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